1 Introduction

A *prime* number is a whole number greater than 1 whose only two whole number factors are 1 and itself. The first few prime numbers are: 2, 3, 5, 7, 11... Any numbers that are not primes are called *composite* numbers (4, 6, 8, 9...).

2 Warm Up

For each of the following numbers, test to see if they are prime. If they are composite, write down their prime factorization (HINT: draw a factor tree).

1. 41:

2. 73:

3. 91:

4. 540:

5. 5040:

3 Sieve of Eratosthenes

Now, we will find all the prime numbers less than 100. The most common way to find these primes is to use the Sieve of Eratosthenes in the table below. At the end, all the circled numbers will be the primes less than 100.

1. First, cross off the number 1. 1 is unique as it is considered neither prime nor composite.
2. Circle the number 2. Now, starting at 4, cross off every multiple of 2 until 100.
3. Circle the number 3. Now, starting at 6, cross off every multiple of 3 until 99. If a number has already been crossed off, you don't need to worry about it.
4. Repeat this step until you have finished crossing off multiples of 11. Now, circle the uncrossed numbers; these are the primes less than 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

4 Infinitely Many Primes?

Euclid was one of the first people to prove the existence of infinitely many primes. Let's take a look to see if we can too! Here are all the prime numbers less than 100:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

1. Find the gaps between the consecutive prime numbers given above. (List them below)

2. As the prime numbers get larger, what gradually happens to the size of the gaps you found above?

3. As we continue going to larger numbers, do you think we can keep finding more prime numbers?

5 Twin Primes

Twin primes are pairs of prime numbers that differ by 2. Use the list of primes in **Section 3** (above) to help you out with the following:

1. Find all the *twin primes* among prime numbers less than 100.

2. A prime number is called an *isolated prime* if it isn't part of a twin pair. Find all the isolated primes among prime numbers less than 100.

3. Euclid was also one of the first people to conjecture that there are an infinite number of twin primes. However, to this day, there is no proof to this yet. Do you think that there are infinitely many twin primes? (No wrong answers!)

6 Proof by Contradiction

We will now follow Euclid's proof step by step to show that there are infinitely many prime numbers. We will do this by arguing through *contradiction*, meaning that if we can prove our assumption is false, then we can prove the opposite to be true. To begin, let's *assume* that there is a finite (having a limit) number of primes. This means we can list all the primes as:

$$p_1,\ p_2,\ p_3,\ ...,\ p_n$$

This means that p_n is the largest prime number. Therefore, all numbers greater than p_n are composite numbers.

1. A number, A, is divisible by all prime numbers. Write an expression for A in terms of $p_1, p_2, ..., p_n$.

2. Write down an expression for $B = A + 1$ in terms of $p_1, p_2, ..., p_n$.

3. Is B divisible by any of the prime numbers $p_1, p_2, p_3, ..., p_n$? (Hint: Find the remainder when you divide B by each of the given prime numbers.)

4. Using your answer above, can we conclude that B is prime? Why or why not?

5. Why does this mean that we got a *contradiction* with our assumption?

6. What is your conclusion? Are there infinitely many primes?